Explicit realizations of static and nonstatic solenoids and conditions for their existence

1993 J. Phys. A: Math. Gen. 26 3279

(http://iopscience.iop.org/0305-4470/26/13/029)

J. Phys. A: Math. Gen. 26 (1993) 3279–3284. Printed in the UK

Explicit realizations of static and non-static solenoids and conditions for their existence

G N Afanasiev, V M Dubovik and S Misicu†

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow District, 141980, Russia

Received 30 April 1992

Abstract. The conditions are found under which the static and non-static charge and current densities generate the electromagnetic field confined to the finite region of space. The prescriptions are given for the construction of static magnetic solenoids of an arbitrary geometrical form.

1. Introduction

Under the solenoids one usually understands specific configurations of charge ρ and current j densities which being confined to the space region S generate electromagnetic strengths E, H vanishing outside S. It is the aim of present consideration to find general expressions for ρ and j satisfying these conditions. The plan of our exposition is as follows. The non-static solenoids are studied in section 2. The current density inside S is parametrized in three different ways. The conditions for the disappearance of E, H outside S are formulated in terms of these parametrizations. The static current configurations confined to the space region S are treated in section 3. The conditions for the vanishing of magnetic field outside S are obtained by means of magnetization formalism. It turns out that the solenoid of the arbitrary geometrical form can be constructed by filling this form with the substance having divergence-free magnetization.

2. Non-static solenoids

Let the charge and current densities be periodical functions of time: $\rho\exp(-\mathrm{i}\omega t)$, $j\exp(-\mathrm{i}\omega t)$. In what follows we omit the factor $\exp(-\mathrm{i}\omega t)$. From the continuity equation it follows that $\mathrm{div}\,j = \mathrm{i}\omega\rho$. We write out the general expansion of the magnetic vector potential (VP) A and the scalar electric potential ϕ (it is valid outside the region where ρ, $j \neq 0$

$$A = \frac{1}{c}4\pi\mathrm{i}k\sum_{lm\tau}A_l^m(\tau)a_l^m(\tau) \tag{2.1}$$

$$\phi = 4\pi\mathrm{i}k\sum h_l Y_l^m q_l^m \qquad \mathrm{div}A + \frac{1}{c}\dot{\phi} = 0. \tag{2.2}$$

† Permanent address: Institute for Atomic Physics, PO Box MG-6, Bucharest, Romania.

The elementary VP $A_l^m(\tau)$ are the vector solutions of the Helmholtz equation. Index τ refers to electric (E), magnetic (M) and longitudinal (L) multipoles. The elementary VP with difference l, m, τ are orthogonal on the sphere of an arbitrary radius. In the differential form they are given by [1–3]

$$A_l^m(L) = \frac{1}{k}\nabla h_l Y_l^m \qquad A_l^m(M) = \frac{1}{\sqrt{l(l+1)}} L h_l Y_l^m$$
$$A_l^m(E) = -\frac{\mathrm{i}}{k}\frac{1}{\sqrt{l(l+1)}}\nabla\times(L h_l Y_l^m). \tag{2.3}$$

Here L is the operator of the orbital angular momentum

$$L = -\mathrm{i}(r\times\nabla) \qquad h_l \equiv h_l(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} H_{l+1/2}^{(1)}(kr) \qquad Y_l^m \equiv Y_l^m(\theta,\varphi)$$

further

$$q_l^m = \int j_l Y_l^{m*}\rho\,\mathrm{d}V \qquad a_l^m(\tau) = \int B_l^{m*} j\,\mathrm{d}V.$$

The vector functions $B_l^m(\tau)$ are obtained from (2.3) by substituting

$$j_l \equiv (\pi/2kr)^{1/2} J_{l+1/2}(kr)$$

instead of $h_l(kr)$. In the expansions (2.1) and (2.2) it was suggested that ρ and j occupy the part of space S which includes the origin while the VP is evaluated at the point r lying outside S. It may happen that S does not contain the origin (e.g. for the toroidal solenoid $(\rho-d)^2+z^2=R^2$). In that case, for r lying between the origin and S, the role of j_l and h_l should be interchanged in (2.1)–(2.3). Now we require the disappearance of $B=\nabla\times A$ outside S. Taking into account that

$$\nabla\times A_l^m(L) = 0 \qquad \nabla\times A_l^m(M) = \mathrm{i}kA_l^m(E) \qquad \nabla\times A_l^m(E) = -\mathrm{i}kA_l^m(M)$$

we obtain (outside S)

$$B = \frac{1}{c}4\pi k^2\sum_{lm}[A_l^m(M)a_l^m(E) - A_l^m(E)a_l^m(M)] = 0.$$

Since $A_l^m(\tau)$ are linear independent and orthogonal

$$a_l^m(E) = a_l^m(M) = 0 \tag{2.4}$$

when $B=0$. Thus, A should be of the form

$$A = \frac{1}{c}4\pi\mathrm{i}\nabla\sum_{lm} h_l Y_l^m a_l^m(L) \tag{2.5}$$

where $a_l^m(L)$ are given by

$$a_l^m(L) = \frac{1}{k}\int\nabla(j_l Y_l^{m*})j\,\mathrm{d}V = \mathrm{i}cq_l^m.$$

It is easy to check that $E = -\nabla\phi - (1/c)\dot{A} = 0$ for the electromagnetic potentials (2.1) and (2.5). Due to the completeness of expansion (2.1) it follows that (2.2) and (2.4) realize the most general non-static solenoid. To clarify the physical meaning of (2.4) we parametrize the current density. The simplest parametrization is [4]:

$$j = \nabla\psi + \nabla \times M \qquad \mathrm{div} M = 0. \tag{2.6}$$

The substitution of (2.6) into (2.4) gives:

$$\int j_l Y_l^{m*}(r \cdot M)\mathrm{d}V = \int j_l Y_l^{m*} r \cdot (\nabla \times M)\mathrm{d}V = 0.$$

The next one is the Helmholtz parametrization [3]

$$j = \nabla f_1 + \nabla \times (r \cdot f_2) + \nabla \times \nabla \times (r \cdot f_3). \tag{2.7}$$

Being inserted into (2.4) it leads to the following conditions on f_2 and f_3

$$\begin{aligned}
&\int j_l Y_l^{m*}\left[2r\frac{\partial f_2}{\partial r} + 6f_2 + k^2r^2 f_2 + r\frac{\partial}{\partial r}\,\mathrm{div}\,(r \cdot f_2)\right]\mathrm{d}V = 0\\
&\int j_l Y_l^{m*}\left[2\mathrm{div}\,(r \cdot f_3) + k^2r^2 f_3 + r\frac{\partial}{\partial r}\,\mathrm{div}\,(r \cdot f_3)\right]\mathrm{d}V = 0.
\end{aligned} \tag{2.8}$$

The last parametrization we consider is:

$$j = \sum_{lm}[r f_{lm}^{(1)} + f_{lm}^{(2)}\nabla + f_{lm}^{(3)}(r \times \nabla)]Y_l^m. \tag{2.9}$$

(An arbitrary vector function can be presented in this form—see e.g. [5].) The functions $f_{lm}^{(i)}$ depend on the radial coordinate only. Making the same procedure as before we get the following conditions on $f_{lm}^{(1,3)}$

$$\int j_l(kr) f_{lm}^{(1)} r^2\,\mathrm{d}r = \int j_l(kr) f_{lm}^{(3)} r^2\,\mathrm{d}r = 0. \tag{2.10}$$

It seems at first (since the Bessel functions form the complete system) that (2.10) may be fulfilled only if $f_{lm}^{(1)} = f_{lm}^{(3)} = 0$. This would be true if (2.10) were satisfied for all values of k. However, for the fixed value of k (as for the treated case) one may always find non-trivial (i.e. non-zero) functions $f_{lm}^{(1,3)}$ meeting (2.10). The particular examples of such current densities may be found in [6, 7]. The point-like realization of non-static solenoids was proposed in [8]. It is obtained when the following choice of charge and current densities is made: $\rho = D\Delta\delta^3(r), j = \mathrm{i}\omega D\nabla\delta^3(r)$ where D is a constant. In [9] the non-static solenoid has been realized by means of cylindrical capacitor. Beautiful experiments with such a capacitor were described in [10]. The reason for treating non-static solenoids is that they emit the waves of electromagnetic potentials propagating with the velocity of light [7, 9, 11, 12]. Since they do not carry the electromagnetic energy ($E = H = 0$ in them), these waves can be detected only at the quantum level. Electromagnetic potentials generated by the non-static densities treated in [8, 9, 12] can be removed by the single-valued gauge transformation and, thus, they are not observable. If the space region S where $\rho, j \neq 0$ is simply connected and includes the origin then the electromagnetic potentials (2.2) and (2.5) outside S may be eliminated by the following gauge transformation

$$A \to A' = A - \nabla\chi \qquad \phi \to \phi' = \phi + \frac{1}{c}\dot{\chi}$$

$$\chi = \frac{1}{c}4\pi \mathrm{i}\sum h_l Y_l^{m*} a_l^m. \tag{2.11}$$

It is single-valued everywhere outside S and as a consequence, the waves of electromagnetic potentials have no physical meaning in this case. The situation changes when space surrounding S is not simply connected. Similarly to the static case there arises a chance to find ρ and j generating electromagnetic potentials that cannot be removed by the single-valued gauge transformation (details may be found in [7]).

3. Static magnetic solenoids

The vector potential corresponding to the static current density equals

$$A = \frac{1}{c}\int G_0 j(r')\,\mathrm{d}V' \qquad G_0 = |r - r'|^{-1}. \tag{3.1}$$

Instead of the current the equivalent magnetization can be used [13, 14]: $j = c\nabla \times M$. It is confined entirely inside the solenoid. Then

$$A = \int G_0 \nabla \times M(r')\,\mathrm{d}V'. \tag{3.2}$$

The magnetic induction is given by

$$B = \nabla \times A = \int G_0 \nabla \times (\nabla \times M)\,\mathrm{d}V' = \int G_0(\mathrm{grad}\,\mathrm{div}\,M - \Delta M)\,\mathrm{d}V. \tag{3.3}$$

Integrating by parts we get

$$B = H + 4\pi M \tag{3.4}$$

where the magnetic field strength equals

$$H = \mathrm{grad}\int G_0(r, r')\,\mathrm{div}\,M(r')\,\mathrm{d}V'. \tag{3.5}$$

Let $\mathrm{div}\,M = 0$. Then $H = 0$ everywhere, while B differs from zero in those space regions where $M \neq 0$ (i.e. inside the solenoid). We conclude: to construct the static magnetic solenoid of an arbitrary geometrical form it is enough to fill this form by the substance having the divergence-free magnetization. The magnetized rings used in the experiments testing the Aharonov–Bohm effect [15] are the particular realizations of such a magnetized filament. The usual toroidal solenoids [16] and their generalizations [17] also correspond to the particular choice of divergence-free magnetization. One should distinguish the real magnetization from the fictitious. Equation (3.2) is valid in either case. Equations (3.3)–(3.5) are related to the real magnetization (i.e. when the space region S is filled with the magnetized substance). On the other hand, if the magnetization is fictitious (i.e. when it is introduced formally using the relation $j = (\nabla \times M)$ then $B = H$ everywhere. Instead of (3.3)–(3.5) one has

$$B = H = \nabla \times A = \operatorname{grad} \int G_0 \operatorname{div} M \, dV' + 4\pi M.$$

The conditions for the disappearance of B outside the spatial region S to which the current j is confined were also obtained in the interesting reference [18] (although in a slightly different context). If j_μ denote the spherical components of j ($j_0 = j_z$, $j_{\pm 1} = \mp(j_x \pm i j_y)/\sqrt{2}$ then conditions found in [18] are

$$R_{-1}^{lm} = \left(\frac{1}{2}\frac{l-m}{l+m+1}\right)^{1/2} R_0^{l,m+1} \qquad R_1^{lm} = \left(\frac{1}{2}\frac{l+m}{l-m+1}\right)^{1/2} R_0^{l,m-1}. \tag{3.6}$$

Here $R_\mu^{lm} = \int r^l Y_l^{m*} j_\mu \, dV$. These equations may serve as checking points verifying the B (or H) disappearance outside S for the chosen current configuration. However, these equations are not constructive in the sense that they do not give the recipe for concrete realization of the current density. In addition, the evaluation of integrals occurring in it is not a trivial task even for the simplest toroidal configurations [19]. On the other hand, the use of the magnetization formalism makes the construction of magnetic solenoids almost trivial. This fact is intuitively used by the experimentalists. The components of the current density corresponding to the chosen magnetization ($j = c\nabla \times M$, $\operatorname{div} M = 0$) satisfy (3.6) automatically. The final remark concerns the self-screened current distributions. Let the current be of the form $j = c\nabla \times \nabla \times t$, $\operatorname{div} t = 0$. Substituting this into (3.1) we obtain $A = 4\pi t(r)$. Thus, the vector potential differs from zero only in those space regions where $t \neq 0$. Such a representation of t takes place, e.g. for the toroidal moments [20]. Physically this means that toroidal moments uniformly distributed along the arbitrary closed curve and tangential to it generate vector potential vanishing outside this curve. For the circular chain of toroidal moments this fact has recently been admitted in [7].

4. Conclusion

We briefly summarize the main results obtained:

(i) The conditions are found for the disappearance of electromagnetic field strengths outside the space region to which the non-static charge and particle densities are confined.

(ii) The prescriptions are given for the construction of static magnetic solenoids of an arbitrary geometrical form.

Acknowledgments

The authors thank the referees for their efforts to improve the contents of the manuscript.

References

[1] Blatt J M and Weisskopf V F 1952 *Theoretical Nuclear Physics* (New York: Wiley) appendix II
[2] Rose M E 1955 *Multipole Fields* (New York: Wiley) ch 2–4
[3] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vol 2 (New York: McGraw-Hill) ch 13
[4] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vol 1 (New York: McGraw-Hill) ch 1
[5] Barrera R G, Estevez G A and Giraldo J 1985 *Eur. J. Phys.* **6** 287
[6] Meyer-Vernet N 1989 *Am. J. Phys.* **57** 1084
Goedecke G 1964 *Phys. Rev.* B **135** 281
Pearle P 1977 *Found. Phys.* **7** 931
Abbott T A and Griffiths D J 1985 *Am. J. Phys.* **53** 1203
Bohm D and Weinstein M 1948 *Phys. Rev.* **74** 1789
[7] Afanasiev G N 1993 *J. Phys. A: Math. Gen.* **26** 731
[8] Miller M A 1986 Izvestiya Vysshikh Uchebnych Zavedeney *Radiofizika* **29** 391
[9] Petuchov V R 1991 *Preprint* ITEP 105–91
[10] Bartlett D F and Gengel G 1989 *Phys. Rev.* **39** 938
[11] Djatlov V L and Lavrentiev M M 1990 *The Waves of Electro-magnetic Potentials* (unpublished)
[12] Petuchov V R 1991 *Preprint* ITEP 106–91
[13] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley) ch 5
[14] Carrascal B, Estevez G A and Lorenzo V 1991 *Amer. J. Phys.* **59** 233
[15] Peshkin M and Tonomura A 1989 *The Aharonov–Bohm Effect* (Berlin: Springer) pp 118–38
Tonomura A 1992 *Adv. Phys.* **41** 59
[16] Afanasiev G N 1987 *J. Comput. Phys.* **69** 196
[17] Afanasiev G N 1993 *Sov. J. Particles and Nuclei* **24** 512
[18] Bosco B and Sacchi M T 1974 *J. Math. Phys.* **15** 225
[19] Afanasiev G N and Dubovik V M 1992 *J. Phys. A: Math. Gen.* **25** 4869
[20] Dubovik V M and Tugushev V V 1990 *Phys. Rep.* **187** 145